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Circular disclination loops in nonlocal elasticity

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Abstract

Solutions for problems of circular twist and wedge disclination loops in an infinitely extended linear isotropic nonlocal elastic medium are obtained assuming an appropriate nonlocal modulus. The equilibrium equation is satisfied by introducing the Kröner's stress function tensor. The Laplace and Hankel transforms are used to obtain the stress fields and the stored elastic energies. The oscillatory integrals containing Bessel functions are transformed into integrands which decay exponentially, thus producing a solution more amenable to numerical quadrature. It is found that maximum stresses are reached at some distance from the defect line. The obtained solutions lead to finite values of stresses at this line and reduce to the classical ones in the long wave-length limit. © 2000 Elsevier Science Ltd. All rights reserved.

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1. Introduction

Disclinations have been receiving the attention of many researchers, in particular, in the context of their applications to polymers (Li and Gilman, 1970), liquid crystals (Bouligand, 1981), biological structures (Harris, 1974), grain boundaries (Li, 1972), amorphic solids (Richter et al., 1984), rotation plastic deformation (Romanov and Vladimirov, 1992) (see also a review article of Romanov and Vladimirov, 1983). Elastic fields and energies of circular twist and wedge disclinations have been investigated in the frame-work of classical elasticity by Li and Gilman (1970), Huang and Mura (1970),

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Liu and Li (1971), Kuo and Mura (1972) and Kuo et al. (1973). For comprehensive review and additional references, see also a study of Kolesnikova and Romanov (1986).

A limitation of the abovementioned solutions is that the stress field has nonphysical singularity at the disclination line and the elastic energy diverges if one does not cut the defect core. Many efforts have been made to improve classical elastic solutions, for instant combining the elastic and discrete approaches for better description of high distorted region near defect. A review of the Frenkel–Kontorova and Pierls–Nabarro models for dislocations and their generalizations was made by Hirth and Lothe (1968). The semi-discrete approach according to which the crystal in the vicinity of defect is treated as a discrete lattice, and for the remainder of the crystal an elastic continuum model is used, has been discussed extensively by Teodosiu (1982) and Duesbery (1989). But all the abovementioned improvements only consider straight dislocations.

Though the literature concerning various models of the dislocation core is very extensive, that for the disclination core is not numerous. We can only mention studies of Doyama and Cotteril (1984) and Mikhailin and Romanov (1986) on computer simulation of straight disclinations. As far as we can judge, no other attempts have been made to improve the situation in the vicinity of the disclination line, especially for circular disclinations. Recently, great advances have been made in crystal defect research by application of nonlocal elasticity which takes into account interatomic long-range forces. Several versions of nonlocal continuum mechanics have been proposed by Kröner (1967), Eringen (1972), Edelen (1976), Kunin (1986) and others. Theory of nonlocal elasticity indicates its power in the study of such problems as line crack (Eringen et al., 1977), rectangular rigid stamp (Artan and Yelkenci, 1996), straight edge (Eringen, 1977a) and screw (Eringen, 1977b; Gao, 1990) dislocations, straight wedge and twist disclinations (Povstenko, 1995a), circular prismatic and glide dislocation loops (Povstenko, 1995b).

In this paper, the solutions for the problems of circular twist disclination loop with the Frank vector normal to the loop plane and wedge disclination loop with the Frank vector in a loop plane are obtained by solving the field equations of nonlocal elasticity proposed by Eringen (1972). We also consider the circular rotation dislocation loop as the solution for a defect of such a type, very similar to the solution for the circular twist disclination loop. The obtained stress fields have no singularities at the defect lines, in contrast to the classical elastic ones and reduce to the classical results in the long wavelength limit. The stored elastic energies are also calculated.

2. The governing equations and statement of the problem

For the static case with vanishing body force, the basic equations for a linear isotropic nonlocal elastic solid are (Eringen, 1972)

$$\nabla \cdot \mathbf{t} = 0, \tag{1}$$

$$\mathbf{t}(\boldsymbol{x};\tau) = \int_V \alpha\big(|\boldsymbol{x}' - \boldsymbol{x}|, \tau\big)\boldsymbol{\sigma}(\boldsymbol{x}')\,\mathrm{d}v(\boldsymbol{x}'), \tag{2}$$

$$\boldsymbol{\sigma}(\boldsymbol{x}') = \lambda\,\mathrm{tr}\,\mathbf{e}(\boldsymbol{x}')\mathbf{I} + 2\mu\mathbf{e}(\boldsymbol{x}'), \tag{3}$$

$$\mathbf{e}(\boldsymbol{x}') = \mathrm{Def}\,\mathbf{u}(\boldsymbol{x}'). \tag{4}$$

Here, $\boldsymbol{x}$ and $\boldsymbol{x}'$ are reference and running points, $\mathbf{t}$ and $\boldsymbol{\sigma}$ the nonlocal and classical stress tensors, $\mathbf{u}$ is the displacement vector, $\mathbf{e}$ the linear strain tensor, $\mathbf{I}$ the unit tensor, λ and μ are the Lamé constants.

The nonlocal modulus $\alpha(|\boldsymbol{x}-\boldsymbol{x}'|, \tau)$ includes the parameter τ proportional to a characteristic length ratio a/l, where a is an internal characteristic length (for example, the lattice parameter) and l is an external characteristic length (Eringen, 1983). This modulus describes the nonlocal interaction, is a delta sequence, and in the classical elasticity limit ($\tau \to 0$) it becomes the Dirac delta function

$$\lim_{\tau \to 0} \alpha(|\boldsymbol{x}-\boldsymbol{x}'|, \tau) = \delta(|\boldsymbol{x}-\boldsymbol{x}'|).$$

Eringen (1972, 1983) has ascertained the properties of nonlocal modulus and found several different forms, giving a perfect match with the atomic lattice dynamics. In the present paper, we employ the following nonlocal modulus

$$\alpha(|\boldsymbol{x}-\boldsymbol{x}'|, t) = \frac{1}{8(\pi t)^{3/2}} \exp\left(-\frac{|\boldsymbol{x}-\boldsymbol{x}'|^2}{4t}\right), \tag{5}$$

where $t = (1/4) l^2 \tau$ (or $t = a^2/4k^2$ with a being the lattice parameter and k an appropriate constant which may be determined either from experiments or by comparison with the results based on lattice dynamics).

The nonlocal modulus (5) is a fundamental solution for the diffusion operator. Thus, the stress field $\boldsymbol{t}$ is determined from the equation

$$\frac{\partial \boldsymbol{t}}{\partial t} - \nabla^2 \mathbf{t} = 0 \tag{6a}$$

under the condition

$$t = 0: \quad \mathbf{t} = \boldsymbol{\sigma}. \tag{6b}$$

Indicating the Laplace transform with respect to t by a superposed bar, we have

$$\nabla^2 \bar{\mathbf{t}} - s\bar{\mathbf{t}} = -\boldsymbol{\sigma}, \tag{6c}$$

where s is the transform variable.

As in classical elasticity (Kröner, 1958), equilibrium equation (1) may be satisfied by introducing the stress function tensor $\boldsymbol{\chi}$

$$\frac{\boldsymbol{t}}{2\mu} = \nabla^2 \boldsymbol{\chi} + \frac{1}{1-\nu}\left[\nabla\nabla(\mathrm{tr}\,\boldsymbol{\chi}) - (\nabla^2 \mathrm{tr}\,\boldsymbol{\chi})\mathbf{I}\right] \tag{7a}$$

subject to

$$\nabla \cdot \boldsymbol{\chi} = 0. \tag{7b}$$

Here, ν is the Poisson ratio. It is easy to show that the following equation is fulfilled

$$(\nabla^2 - s)\nabla^4 \bar{\boldsymbol{\chi}} = -\boldsymbol{\eta}, \tag{8}$$

where $\boldsymbol{\eta}$ is the incompatibility tensor related to the plastic strain $\mathbf{e}^{\mathrm{p}}$ by

$$\boldsymbol{\eta} = \mathrm{Inc}\,\mathbf{e}^{\mathrm{p}} = -\nabla \times \mathbf{e}^{\mathrm{p}} \times \nabla \tag{9a}$$

or to the dislocation density tensor $\boldsymbol{\alpha}$ by

$$\boldsymbol{\eta} = (\boldsymbol{\alpha} \times \nabla)^{\mathrm{s}}, \tag{9b}$$

where the superscript s indicates symmetrization.

Defect of the Volterra type can be formed by cutting a surface S in a body and the subsequent rigid relative translation with the Burgers vector $\mathbf{b}$ and the rotation with the Frank vector $\boldsymbol{\Omega}$ of two sides of the cut. The corresponding plastic distortion $\boldsymbol{\beta}^{\mathrm{p}}$ has the form obtained by Mura (1972)

$$\boldsymbol{\beta}^{\mathrm{p}} = -\delta(\mathbf{S})(\mathbf{b} + \boldsymbol{\Omega} \times \mathbf{r}), \tag{10}$$

where $\delta(\mathbf{S})$ is the vector delta-function lumped on the surface S (Kunin, 1986).

3. Circular twist disclination loop

Consider a loop of radius R in the plane $z = 0$ of cylindrical coordinates r, ϑ, z centered at the origin. A twist disclination is described by the Frank vector in the z direction Ω_z. The only nonzero component of the plastic distortion tensor is expressed by

$$\beta^{\mathrm{p}}_{z\vartheta} = -\Omega_z r H(R - r)\delta(z), \tag{11}$$

where $H(x)$ is the Heaviside function and $\delta(z)$ is the Dirac delta function.

The nonzero components of the incompatibility tensor are given by

$$\eta_{r\vartheta} = \frac{1}{2}\Omega_z R\delta(r - R)\delta'(z), \tag{12a}$$

$$\eta_{\vartheta z} = -\frac{1}{2}\Omega_z R\left[\delta'(r - R) + \frac{2}{R}\delta(r - R)\right]\delta(z). \tag{12b}$$

Using the component representation of Laplacian of the symmetrical second-order tensor in cylindrical coordinates, Eq. (8) reads

$$\left(\frac{\mathrm{d}^2}{\mathrm{d}z^2} - \xi^2 - s\right)\left(\frac{\mathrm{d}^2}{\mathrm{d}z^2} - \xi^2\right)^2 \bar{\chi}^{(*1)}_{\vartheta z} = M_z \xi\delta(z), \tag{13a}$$

$$\left(\frac{\mathrm{d}^2}{\mathrm{d}z^2} - \xi^2 - s\right)\left(\frac{\mathrm{d}^2}{\mathrm{d}z^2} - \xi^2\right)^2 \bar{\chi}^{(*2)}_{r\vartheta} = -M_z\delta'(z), \tag{13b}$$

where $M_z = (1/2)\Omega_z R^2 J_2(R\xi)$, J_n is the Bessel function of the first kind of order n, an asterisk with a superscript in parentheses denote the Hankel transform of the corresponding order with ξ being the transform variable.

We look for the solution of Eqs. (13) bounded at infinity in the form

$$\bar{\chi}^{(*1)}_{\vartheta z} = (A_1 + B_1\xi|z|)\exp(-\xi|z|) + C_1 \exp\left[-\left(\xi^2 + s\right)^{1/2}|z|\right], \tag{14a}$$

$$\bar{\chi}^{(*2)}_{r\vartheta} = \left\{(A_2 + B_2\xi|z|)\exp(-\xi|z|) + C_2 \exp\left[-\left(\xi^2 + s\right)^{1/2}|z|\right]\right\}\operatorname{sign} z. \tag{14b}$$

Unknown coefficients are found to be:

$$A_1 = -\frac{M_z}{2}\left(\frac{1}{2\xi^2 s} - \frac{1}{s^2}\right), \quad B_1 = B_2 = -\frac{M_z}{4\xi^2 s},$$

$$C_1 = -\frac{M_z \xi}{2s^2}\left(\xi^2 + s\right)^{-1/2}, \quad A_2 = -C_2 = \frac{M_z}{2s^2}.$$

Inverting the Laplace and Hankel transforms, we have from Eqs. (14)

$$\chi_{\vartheta z} = -\frac{1}{8}\Omega_z R^2 \int_0^\infty \frac{1}{\xi} J_2(R\xi) J_1(r\xi) Q(\xi, |z|, t)\, \mathrm{d}\xi, \tag{15a}$$

$$\chi_{r\vartheta} = -\frac{1}{8}\Omega_z R^2 \operatorname{sign} z \int_0^\infty \frac{1}{\xi} J_2(R\xi) J_2(r\xi) U(\xi, |z|, t)\, \mathrm{d}\xi, \tag{15b}$$

and from Eq. (7a)

$$t_{r\vartheta} = -\frac{1}{2} A_z \operatorname{sign} z \int_0^\infty J_2(R\xi) J_2(r\xi) S(\xi, |z|, t)\xi\, \mathrm{d}\xi, \tag{16a}$$

$$t_{\vartheta z} = \frac{1}{2} A_z \int_0^\infty J_2(R\xi) J_1(r\xi) T(\xi, |z|, t)\xi\, \mathrm{d}\xi, \tag{16b}$$

where

$$Q(\xi, |z|, t) = \frac{2\xi^3}{\sqrt{\pi}} \int_t^\infty \zeta^{-1/2}(\zeta - t) P(\xi, |z|, \zeta)\, \mathrm{d}\zeta, \tag{17a}$$

$$S(\xi, |z|, t) = -\frac{|z|}{\sqrt{\pi}} \int_t^\infty \zeta^{-3/2} P(\xi, |z|, \zeta)\, \mathrm{d}\zeta, \tag{17b}$$

$$T(\xi, |z|, t) = \frac{2\xi}{\sqrt{\pi}} \int_t^\infty \zeta^{-1/2} P(\xi, |z|, \zeta)\, \mathrm{d}\zeta, \tag{17c}$$

$$U(\xi, |z|, t) = \frac{\xi^2 |z|}{\sqrt{\pi}} \int_t^\infty \zeta^{-3/2}(\zeta - t) P(\xi, |z|, \zeta)\, \mathrm{d}\zeta, \tag{17d}$$

$$P(\xi, |z|, \zeta) = \exp\left(-\zeta\xi^2 - \frac{z^2}{4\zeta}\right), \qquad A_z = \frac{1}{2}\mu R^2 \Omega_z.$$

Changing integrals over ζ and ξ, we can transform the oscillatory integrals containing Bessel functions into integrands which decay exponentially, thus producing a solution more amenable to numerical quadrature

$$t_{r\vartheta} = \frac{z}{2\sqrt{\pi}} A_z \int_t^\infty \zeta^{-3/2} \exp\left(-\frac{z^2}{4\zeta}\right) F(2, 2; 1)\, \mathrm{d}\zeta, \tag{18a}$$

$$t_{\vartheta z} = \frac{1}{\sqrt{\pi}} A_z \int_t^\infty \zeta^{-1/2} \exp\left(-\frac{z^2}{4\zeta} \right) F(2, 1; 2)\, \mathrm{d}\zeta, \tag{18b}$$

where

$$F(a, b; c) = \int_0^\infty \exp\left(-\xi^2 \zeta \right) J_a(R\xi) J_b(r\xi) \xi^c\, \mathrm{d}\xi. \tag{19}$$

Integrals $F(a, b; c)$ used in this paper are expressed in terms of modified Bessel functions and listed in Appendix A.

Passage in Eqs. (18) to the limit $t \to 0$ leads to the local elasticity solution

$$\sigma_{r\vartheta} = A_z \operatorname{sign} z\, I(2, 2; 1), \tag{20a}$$

$$\sigma_{\vartheta z} = A_z I(2, 1; 1) \tag{20b}$$

with the standard notation for the integrals of Lipschitz–Hankel type

$$I(a, b; c) = \int_0^\infty \exp(-\xi |z|) J_a(R\xi) J_b(r\xi) \xi^c\, \mathrm{d}\xi, \tag{21}$$

investigated by Eason et al. (1955), Kuo and Mura (1972), Salamon and Walter (1979) and Hanson and Wang (1997) and related to the complete elliptic integrals.

Eqs. (20) coincide with those presented by Kolesnikova and Romanov (1986), however $\sigma_{r\vartheta}$ differs from the one obtained by Kuo et al. (1973). An expression for the displacement component u_ϑ in the last-mentioned paper is correct, but for the stress component $\sigma_{r\vartheta}$ has an error (it can be easy verified by differentiation).

In the classical theory, the general expression for the energy of twist disclination loop

$$E^{\mathrm{T}} = \frac{1}{2} \Omega_z \int_0^{2\pi} \int_0^{R-R_c} \sigma_{\vartheta z}|_{z=0} r^2\, \mathrm{d}r\, \mathrm{d}\vartheta \tag{22}$$

after substituting $\sigma_{\vartheta z}$ from Eq. (20b) yields

$$E^{\mathrm{T}} = \pi A_z \Omega_z R^2 I(2, 2; 0)|_{\substack{r = R - R_c \\ z = 0}}. \tag{23}$$

In Eq. (22), the upper limit of integration R is replaced by $R - R_c$, where R_c is the core radius, as for classical solution the energy diverges due to the stress singularity at the disclination line.

The energy of a loop in nonlocal elastic solid reads

$$E^{\mathrm{T}} = \sqrt{\pi} A_z \Omega_z R^2 \int_t^\infty \zeta^{-1/2} F(2, 2; 1)|_{r=R}\, \mathrm{d}\zeta. \tag{24}$$

4. Circular rotation dislocation loop

For a circular rotation dislocation loop described by the Burgers vector b_ϑ, the dislocation density tensor $\boldsymbol{\alpha}$ has the only nonzero component

$$\alpha_{\vartheta\vartheta} = b_\vartheta \delta(r - R)\delta(z). \tag{25}$$

Eq. (9b) gives the nonzero components of the incompatibility tensor

$$\eta_{r\vartheta} = \frac{1}{2} b_\vartheta \delta(r - R)\delta'(z), \tag{26a}$$

$$\eta_{\vartheta z} = -\frac{1}{2} b_\vartheta \left[\delta'(r - R) + \frac{1}{R}\delta(r - R) \right] \delta(z). \tag{26b}$$

If we substitute $\Omega_z R$ by b_ϑ, Eqs. (12) coincide with Eqs. (26), except a factor 2 in the second term of the right-hand side of Eq. (26b). It is obvious that the solution for the circular rotation dislocation loop will be very similar to the solution for the circular twist disclination loop. We briefly describe the corresponding results.

The nonzero components of the stress function tensor are found to be

$$\chi_{\vartheta z} = -\frac{1}{8} b_\vartheta R \int_0^\infty \frac{1}{\xi} \left[J_2(R\xi) - \frac{1}{R\xi} J_1(R\xi) \right] J_1(r\xi) Q(\xi, |z|, t)\, \mathrm{d}\xi, \tag{27a}$$

$$\chi_{r\vartheta} = -\frac{1}{8} b_\vartheta R \operatorname{sign} z \int_0^\infty \frac{1}{\xi} J_2(R\xi) J_2(r\xi) U(\xi, |z|, t)\, \mathrm{d}\xi, \tag{27b}$$

while the nonlocal stress tensor components have the following form

$$t_{r\vartheta} = -\frac{1}{2} A_\vartheta \operatorname{sign} z \int_0^\infty J_2(R\xi) J_2(r\xi) S(\xi, |z|, t)\xi\, \mathrm{d}\xi, \tag{28a}$$

$$t_{\vartheta z} = \frac{1}{2} A_\vartheta \int_0^\infty \left[J_2(R\xi) - \frac{1}{R\xi} J_1(R\xi) \right] J_1(r\xi) T(\xi, |z|, t)\xi\, \mathrm{d}\xi \tag{28b}$$

or (after changing integrals over ζ and ξ)

$$t_{r\vartheta} = \frac{z}{2\sqrt{\pi}} A_\vartheta \int_t^\infty \zeta^{-3/2} \exp\left(-\frac{z^2}{4\zeta} \right) F(2, 2; 1)\, \mathrm{d}\zeta, \tag{29a}$$

$$t_{\vartheta z} = \frac{1}{\sqrt{\pi}} A_\vartheta \int_t^\infty \zeta^{-1/2} \exp\left(-\frac{z^2}{4\zeta} \right) \left[F(2, 1; 2) - \frac{1}{R} F(1, 1; 1) \right] \mathrm{d}\zeta, \tag{29b}$$

where $A_\vartheta = (1/2)\mu R b_\vartheta$.

Proceeding to the limit $t \longrightarrow 0$ gives

$$\sigma_{r\vartheta} = A_\vartheta \operatorname{sign} z\, I(2, 2; 1), \tag{30a}$$

$$\sigma_{\vartheta z} = A_\vartheta \left[I(2, 1; 1) - \frac{1}{R} I(1, 1; 0) \right]. \tag{30b}$$

The energy of circular rotation dislocation loop is expressed by

$$E^{\mathrm{R}} = \pi A_\vartheta b_\vartheta R \left[I(2,2;0)|_{\substack{r=R-R_c \\ z=0}} - \frac{1}{R} I(1,2;-1)|_{\substack{r=R-R_c \\ z=0}} \right], \tag{31}$$

$$E^{\mathrm{R}} = \sqrt{\pi} A_\vartheta b_\vartheta R \int_t^\infty \zeta^{-1/2} \left[F(2,2;1)|_{r=R} - \frac{1}{R} F(1,2;0)|_{r=R} \right] \mathrm{d}\zeta \tag{32}$$

for local and nonlocal solutions, respectively.

5. Circular wedge disclination loop

For a circular wedge disclination loop with the Frank vector in the plane of the loop (say, in the y direction), the nonzero components of the plastic distortion tensor is

$$\beta^{\mathrm{p}}_{zz} = \Omega_y r H(R-r)\delta(z)\cos\vartheta. \tag{33}$$

The components of the incompatibility tensor are expressed by

$$\eta_{rr} = -\Omega_y \delta(r-R)\delta(z)\cos\vartheta, \tag{34a}$$

$$\eta_{\vartheta\vartheta} = -\Omega_y \left[\delta(r-R) + R\delta'(r-R)\right]\delta(z)\cos\vartheta, \tag{34b}$$

$$\eta_{r\vartheta} = -\Omega_y \delta(r-R)\delta(z)\sin\vartheta, \tag{34c}$$

$$\eta_{zz} = 0, \quad \eta_{rz} = 0, \quad \eta_{r\vartheta} = 0. \tag{34d}$$

Let the components of the stress function tensor $\boldsymbol{\chi}$ be

$$\chi_{rr} = X_{rr}\cos\vartheta, \quad \chi_{\vartheta\vartheta} = X_{\vartheta\vartheta}\cos\vartheta, \quad \chi_{r\vartheta} = X_{r\vartheta}\sin\vartheta.$$

Then, from Eq. (8), we have $X_{rr} = X_{r\vartheta}$ and

$$\left(\frac{\mathrm{d}^2}{\mathrm{d}z^2} - \xi^2 - s\right)\left(\frac{\mathrm{d}^2}{\mathrm{d}z^2} - \xi^2\right)^2 \bar{\Phi} = 2M_y \xi \delta(z), \tag{35}$$

where $M_y = (1/2)\Omega_y R^2 J_2(R\xi)$ and $\bar{\Phi}$ should be replaced by two auxiliary functions $\bar{X}^{(*1)}_{rr} + \bar{X}^{(*1)}_{\vartheta\vartheta}$ and $3\bar{X}^{(*3)}_{rr} - \bar{X}^{(*3)}_{\vartheta\vartheta}$.

Using the recurrence relations for Bessel functions, the stress function components are expressed as

$$\chi_{rr} = -\frac{1}{4}\Omega_y R^2 \int_0^\infty \frac{1}{r\xi^2} J_2(R\xi) J_2(r\xi) Q(\xi, |z|, t)\, \mathrm{d}\xi \cos\vartheta, \tag{36a}$$

$$\chi_{\vartheta\vartheta} = -\frac{1}{4}\Omega_y R^2 \int_0^\infty \frac{1}{\xi} J_2(R\xi) \left[J_1(r\xi) - \frac{1}{r\xi} J_2(r\xi) \right] Q(\xi, |z|, t)\, \mathrm{d}\xi \sin\vartheta, \tag{36b}$$

$$\chi_{r\vartheta} = -\frac{1}{4}\Omega_y R^2 \int_0^\infty \frac{1}{r\xi^2} J_2(R\xi) J_2(r\xi) Q(\xi, |z|, t)\, \mathrm{d}\xi \sin\vartheta. \tag{36c}$$

Eqs. (7a) and (36) permit us to obtain the stress field

$$t_{rr} = A_y \int_0^\infty J_2(R\xi)\Big\{J_1(r\xi)\big[Q(\xi, |z|, t) - T(\xi, |z|, t)\big] - \frac{1}{r\xi}J_2(r\xi)\big[Q(\xi, |z|, t) - (1-\nu)T(\xi, |z|, t)\big]\Big\}\xi \, \mathrm{d}\xi \cos\vartheta, \tag{37a}$$

$$t_{\vartheta\vartheta} = A_y \int_0^\infty J_2(R\xi)\Big\{-\nu J_1(r\xi)T(\xi, |z|, t) + \frac{1}{r\xi}J_2(r\xi)\big[Q(\xi, |z|, t) - (1-\nu)T(\xi, |z|, t)\big]\Big\}\xi \, \mathrm{d}\xi \cos\vartheta, \tag{37b}$$

$$t_{zz} = -A_y \int_0^\infty J_2(R\xi)J_1(r\xi)Q(\xi, |z|, t)\xi \, \mathrm{d}\xi \cos\vartheta, \tag{37c}$$

$$t_{r\vartheta} = A_y \int_0^\infty \frac{1}{r}J_2(R\xi)J_2(r\xi)\big[(1-\nu)T(\xi, |z|, t) - Q(\xi, |z|, t)\big] \, \mathrm{d}\xi \sin\vartheta, \tag{37d}$$

$$t_{rz} = A_y \operatorname{sign} z \int_0^\infty J_2(R\xi)\Big[J_0(r\xi) - \frac{1}{r\xi}J_1(r\xi)\Big]U(\xi, |z|, t)\xi \, \mathrm{d}\xi \cos\vartheta, \tag{37e}$$

$$t_{\vartheta z} = -A_y \operatorname{sign} z \int_0^\infty \frac{1}{r}J_2(R\xi)J_1(r\xi)U(\xi, |z|, t) \, \mathrm{d}\xi \sin\vartheta \tag{37f}$$

with $A_y = \mu R^2 \Omega_y / [2(1-\nu)]$.

The analogue of Eqs. (18) reads

$$t_{rr} = \frac{2}{\sqrt{\pi}}A_y \int_t^\infty \zeta^{-1/2} \exp\Big(-\frac{z^2}{4\zeta}\Big)\Big[(\zeta - t)F(2, 1; 4) - F(2, 1; 2) - \frac{\zeta - t}{r}F(2, 2; 3) + \frac{1-\nu}{r}F(2, 2; 1)\Big] \, \mathrm{d}\zeta \cos\vartheta, \tag{38a}$$

$$t_{\vartheta\vartheta} = \frac{2}{\sqrt{\pi}}A_y \int_t^\infty \zeta^{-1/2} \exp\Big(-\frac{z^2}{4\zeta}\Big)\Big[-\nu F(2, 1; 2) + \frac{\zeta - t}{r}F(2,2;3) - \frac{1-\nu}{r}F(2 ,2; 1)\Big] \, \mathrm{d}\zeta \cos\vartheta, \tag{38b}$$

$$t_{zz} = -\frac{2}{\sqrt{\pi}}A_y \int_t^\infty \zeta^{-1/2}(\zeta - t) \exp\Big(-\frac{z^2}{4\zeta}\Big)F(2, 1; 4) \, \mathrm{d}\zeta \cos\vartheta, \tag{38c}$$

$$t_{r\vartheta} = \frac{2}{\sqrt{\pi}}A_y \int_t^\infty \zeta^{-1/2} \exp\Big(\frac{-z^2}{4\zeta}\Big)\Big[\frac{1-\nu}{r}F(2, 2; 1) - \frac{\zeta - t}{r}F(2, 2; 3)\Big] \, \mathrm{d}\zeta \sin\vartheta, \tag{38d}$$

$$t_{rz} = \frac{z}{\sqrt{\pi}}A_y \int_t^\infty \zeta^{-3/2}(\zeta - t) \exp\Big(-\frac{z^2}{4\zeta}\Big)\Big[F(2, 0; 3) - \frac{1}{r}F(2, 1; 2)\Big] \, \mathrm{d}\zeta \cos\vartheta, \tag{38e}$$

$$t_{\vartheta z} = -\frac{z}{\sqrt{\pi} r} A_y \int_t^\infty \zeta^{-3/2} (\zeta - t) \exp\left(-\frac{z^2}{4\zeta} \right) F(2, 1; 2) \, d\zeta \sin \vartheta. \tag{38f}$$

For completeness, we also present the local solution obtained from Eqs. (37) proceeding to the limit $t \to 0$

$$\sigma_{rr} = A_y \left[-I(2, 1; 1) + |z| I(2, 1; 2) + \frac{1 - 2\nu}{r} I(2, 2; 0) - \frac{|z|}{r} I(2, 2; 1) \right] \cos \vartheta, \tag{39a}$$

$$\sigma_{\vartheta\vartheta} = A_y \left[-2\nu I(2, 1; 1) - \frac{1 - 2\nu}{r} I(2, 2; 0) + \frac{|z|}{r} I(2, 2; 1) \right] \cos \vartheta, \tag{39b}$$

$$\sigma_{zz} = -A_y \left[I(2, 1; 1) + |z| I(2, 1; 2) \right] \cos \vartheta, \tag{39c}$$

$$\sigma_{r\vartheta} = A_y \left[\frac{1 - 2\nu}{r} I(2, 2; 0) - \frac{|z|}{r} I(2, 2; 1) \right] \sin \vartheta, \tag{39d}$$

$$\sigma_{rz} = A_y z \left[I(2, 0; 2) - \frac{1}{r} I(2, 1; 1) \right] \cos \vartheta, \tag{39e}$$

$$\sigma_{\vartheta z} = -A_y \frac{|z|}{r} I(2, 1; 1) \sin \vartheta. \tag{39f}$$

Eqs. (39) coincide with corresponding equations of Kuo and Mura (1972) and Kolesnikova and Romanov (1986), excluding misprints in the signs reversed to those at $I(2, 2; 0)$ and $I(2, 2; 1)$ in the expressions for σ_{rr} obtained by Kuo and Mura. These signs can be verified using Kröner's representation of the classical stress tensor σ in terms of the stress function tensor $\boldsymbol{\chi}$ and equality $\mathrm{tr}\, \boldsymbol{\sigma} = -2\mu(1+\nu)(1-\nu)^{-1} \nabla^2 \, \mathrm{tr}\, \boldsymbol{\chi}$.

In the classical theory, the energy of wedge disclination loop is calculated as

$$E^{\mathrm{W}} = -\frac{1}{2} \Omega_y \int_0^{2\pi} \int_0^{R - R_c} \sigma_{zz}|_{z=0} r^2 \, dr \cos \vartheta \, d\vartheta. \tag{40}$$

Eqs. (39c) and (40) lead to (Kuo and Mura, 1972; Romanov and Vladimirov, 1992)

$$E^{\mathrm{W}} = \frac{\pi}{2} A_y \Omega_y R^2 I(2, 2; 0)|_{\substack{r = R - R_c \\ z = 0}} . \tag{41}$$

The corresponding expression for the disclination loop in the nonlocal material reads

$$E^{\mathrm{W}} = \frac{1}{2} \sqrt{\pi} A_y \Omega_y R^2 \int_t^\infty \zeta^{-3/2} (\zeta + t) F(2, 2; 1)|_{r=R} \, d\zeta. \tag{42}$$

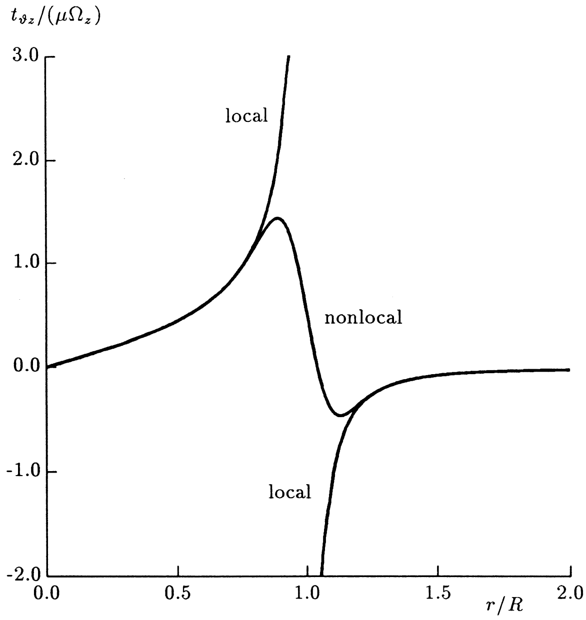

Fig. 1. Shear stress in the plane $z = 0$ for the twist dislocation loop.

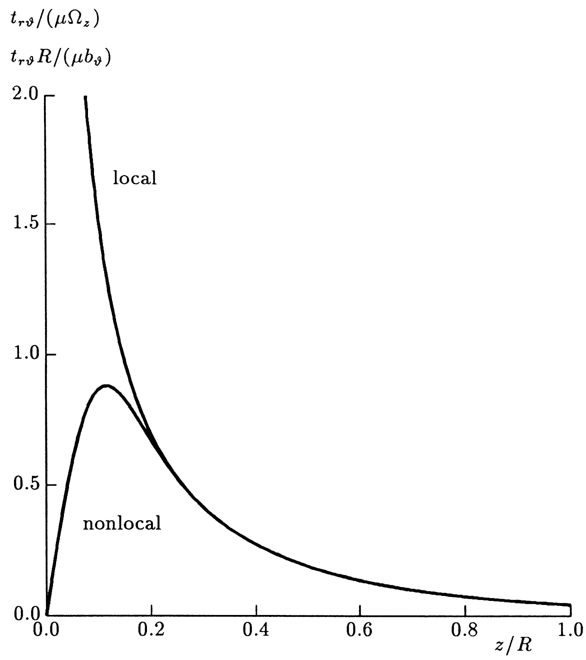

Fig. 2. Shear stress on the slip cylinder $r = R$ for the twist disclination loop or the rotation dislocation loop.

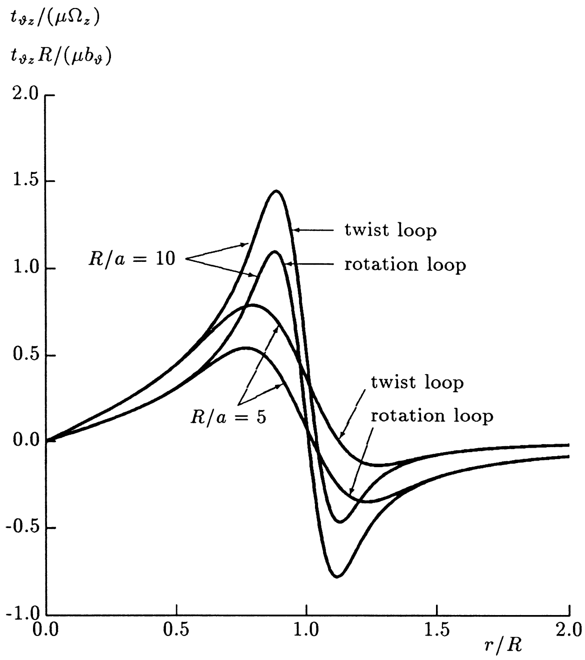

Fig. 3. Shear stress in the loop plane $z = 0$ for various radii of loops.

6. Numerical results and discussion

The nondimensional shear stress component in the plane of the twist disclination loop $z = 0$, $t_{\vartheta z}/(\mu\Omega_z)$ vs. r, calculated from Eq. (18b) is illustrated in Fig. 1 for the loop radius $R = 10a$. Shear stress on the cylinder $r = R$, $t_{r\vartheta}/(\mu\Omega_z)$ vs. z, is shown in Fig. 2. A comparison between the corresponding results for the twist disclination loop and the rotation dislocation loop is presented in Fig. 3 for $R/a = 5$ and 10. In calculations, we have taken $k = 0.94$ (Eringen, 1977b; Gao, 1990). Outside the defect core region, the nonlocal and local curves coincide. Contrary to the classical elasticity, nonlocal stresses are finite at the defect line. From the mathematical standpoint, it follows from regularity of $F(a, b; c)$ for all values of r including $r = R$ if $\tau > 0$ (see Appendix A) and regularity of the integrals between the limits t and infinity in expressions (18) (also Eqs. (29) and (38)) for $t > 0$ (all the singularities in these expressions appear in the long wavelength limit $t \to 0$).

Each component of the stress tensor reaches a maximum at some distance from the defect line. For example, in a case of the twist disclination loops with the radii $R = 5a$ and $10a$, we have, respectively,

Table 1
Energies of twist, rotation and wedge loops

R/a	$\frac{E^{\mathrm{T}}}{\mu\Omega_z^2 R^3}$	$\frac{E^{\mathrm{R}}}{\mu b_\vartheta^2 R}$	$\frac{E^{\mathrm{W}}}{\mu\Omega_y^2 R^3}$
5.0	0.6350	0.3875	0.6337
10.0	0.9736	0.6852	0.9074
20.0	1.3179	1.0071	1.1726
50.0	1.8209	1.4500	1.5526

$t_{\vartheta z}^{\max} \simeq 0.788\mu\Omega_z$ and $t_{\vartheta z}^{\max} \simeq 1.44\mu\Omega_z$ in the loop plane at the distances from the defect line $\simeq 0.2R$ and $\simeq 0.11R$ inside the loop and $t_{\vartheta z}^{\min} \simeq -0.138\mu\Omega_z$ and $t_{\vartheta z}^{\min} \simeq -0.465\mu\Omega_z$ at the distances $\simeq 0.28R$ and $\simeq 0.13R$ outside the loop. For the rotation dislocation loop, the corresponding results read: $t_{\vartheta z}^{\max} \simeq 0.543\mu b_\vartheta/R$ and $t_{\vartheta z}^{\min} \simeq -0.348\mu b_\vartheta/R$ at the distances from the defect line $\simeq 0.23R$ for $R/a = 5$ and $t_{\vartheta z}^{\max} \simeq 1.09\mu b_\vartheta/R$ and $t_{\vartheta z}^{\min} \simeq -0.78\mu b_\vartheta/R$ at the distances $\simeq 0.12R$ for $R/a = 10$. On the slip cylinder, $r = R = 5a$, and we also obtain that $t_{r\vartheta}^{\max} \simeq 0.377\mu\Omega_z$ at a distance $\simeq 0.22R$ from the loop plane and in a case of the loop with $r = R = 10a$ $t_{r\vartheta}^{\max} \simeq 0.879\mu\Omega_z$ at a distance $\simeq 0.12R$. Table 1 shows the energy dependence of three types of loops on the loop radius (for the wedge disclination loop, we have taken $\nu = 1/3$).

Thus, the nonlocal theory eliminates the nonphysical singularities in the stress fields and elastic energies of defects.

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Appendix A

In the case under consideration, integrals

$$F(a, b; c) = \int_0^\infty \exp\left(-\xi^2\zeta\right) J_a(R\xi) J_b(r\xi) \xi^c \, \mathrm{d}\xi$$

are expressed in terms of modified Bessel functions I_n:

$$F(1, 1; 1) = \frac{1}{2\zeta}\exp\left[-\left(p^2+q^2\right)\right] I_1(2pq),$$

$$F(2, 2; 1) = \frac{1}{2\zeta}\exp\left[-\left(p^2+q^2\right)\right] I_2(2pq),$$

$$F(2, 1; 2) = \frac{1}{R\zeta}\exp\left[-\left(p^2+q^2\right)\right]\left[\left(1+q^2\right) I_1(2pq) - pq I_0(2pq)\right],$$

$$F(2, 2; 3) = \frac{1}{\zeta^2}\exp\left[-\left(p^2+q^2\right)\right]\left[pq I_1(2pq) - \frac{1}{2}\left(1+p^2+q^2\right) I_2(2pq)\right],$$

$$F(2, 0; 3) = \frac{1}{2\zeta^2}\exp\left[-\left(p^2+q^2\right)\right]\left[\left(p^2+q^2\right) I_0(2pq) - \left(\frac{p}{q} + 2pq\right) I_1(2pq)\right],$$

$$F(2, 1; 4) = \frac{1}{R\zeta^2}\exp\left[-\left(p^2+q^2\right)\right]\left\{pq\left(p^2+3q^2\right) I_0(2pq) - \left[p^2 + q^2\left(q^2+3p^2\right)\right] I_1(2pq)\right\},$$

where $p = r/(2\sqrt{\zeta})$, $q = R/(2\sqrt{\zeta})$.

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